SPECTRAL-GAP BOUNDS FOR HIGHER-ALPHABET AND WEIGHTED HAMMING GRAPHS

OpenAI 5.6 Sol
GloGlo Research Team

Abstract. Reeves et al. introduced bricklayer graphs as lexicographic initial segments of Hamming graphs. They proved that the adjacency spectral radius of the binary bricklayer graph on n vertices is at most $\log_2 n$, and conjectured the higher-alphabet bound

$$\lambda_{n,a} \le (a-1)\log_a n.$$

We prove the conjecture. In fact, a stronger statement holds: every induced n-vertex subgraph of the minimal-dimensional Hamming graph $H_{\lceil \log_a n \rceil, a}$ has adjacency spectral radius at most $(a-1)\log_a n$. The proof combines the known spectral gap a of a Hamming graph with a support-size bound on the constant component of a Rayleigh maximizer and a scalar concavity inequality. Equality holds at powers of the alphabet size; the inequality is strict otherwise. We then allow substitution rates to vary by coordinate. If $A_w = \sum_j w_j A_j$, where A_j changes the jth coordinate and $w_j > 0$, then every n-vertex support in $H_{d,a}$ with $a^{d-1} \le n \le a^d$ satisfies

$$\rho(A_w[S]) \le (a-1)\sum_j w_j + (a-1)w_{\min}\log_a(n/a^d).$$

At density $1/a$, equality occurs exactly for a one-symbol slice in a minimum-weight coordinate. An induced star shows that this logarithmic form does not extend to arbitrary sparse supports.

1 Introduction

Let

$$H_{d,a} = \underbrace{K_a \,\square\, \cdots \,\square\, K_a}_{d \text{ factors}}$$

be the Hamming graph on words of length d over an alphabet of size a. Two words are adjacent when they differ in exactly one coordinate. Label the words by their base-a values and let $G_{n,a}$ be the subgraph induced by labels $0, \dots, n-1$. Reeves et al. [1] called these *bricklayer graphs*. We study their adjacency spectral radius and coordinate-weighted analogues.

Reeves et al. proved $\lambda_{n,2} \le \log_2 n$ and conjectured

$$\lambda_{n,a} \le (a-1)\log_a n \tag{1}$$

Date: August 30, 2026.
2020 Mathematics Subject Classification. 05C50, 05C35, 60J27.
Key words and phrases. Hamming graph, spectral radius, induced subgraph, spectral gap, coordinate-weighted adjacency operator.
OpenAI 5.6 Sol developed the mathematical argument and performed the accompanying checks. The GloGlo research team posed the problem, established the research direction, and guided the investigation.

for every alphabet size $a \geq 2$. The main theorem proves Eq. (1) and strengthens it from the specified initial segment to every induced subgraph in the smallest Hamming dimension capable of containing n vertices.

The weighted theorem below allows the coordinate weights to vary while retaining symmetry among the $a-1$ edges within each coordinate fibre. This gives an anisotropic extension of the unweighted Hamming-graph bound.

2 A support–spectral-gap lemma

Write $\rho(M)$ for the largest eigenvalue of a real symmetric matrix M.

Lemma 1 (induced-subgraph bound)**.** *Let A be a real symmetric matrix with nonnegative off-diagonal entries and constant row sum D on a set of N states. Suppose every eigenvalue on the orthogonal complement of $\mathbf{1}$ is at most $D-g$, where $g>0$. If S is any set of n states, then*

$$\rho\big(A[S]\big) \leq D - g + g\frac{n}{N}.$$

Proof. Let x be a unit Rayleigh maximizer for $A[S]$, extended by zero to all states. With $u = N^{-1/2}\mathbf{1}$, write $x = \alpha u + z$, where $z \perp u$. Since x has support of size at most n, Cauchy–Schwarz gives

$$\alpha^2 = \frac{(\mathbf{1}^T x)^2}{N} \leq \frac{n}{N}.$$

The spectral decomposition of A therefore yields

$$x^T A x \leq D\alpha^2 + (D-g)(1-\alpha^2) \leq D - g + g\frac{n}{N}.$$

Maximizing over vectors supported on S proves the claim. □

The lemma is a standard Rayleigh-quotient consequence of a spectral gap. Its application below, rather than the lemma itself, is the contribution of this article. Closely related support-versus-gap and Dirichlet-eigenvalue inequalities are standard for reversible Markov operators; see, for example, Chen [3].

3 The higher-alphabet bound

The adjacency eigenvalues of $H_{d,a}$ are

$$d(a-1) - aj, \qquad j = 0, \ldots, d.$$

Thus $H_{d,a}$ is $D = d(a-1)$ regular, has $N = a^d$ vertices, and has adjacency spectral gap $g = a$.

Theorem 1 (minimal-dimension Hamming bound)**.** *Fix integers $a \geq 2$ and $n \geq 2$, and put $d = \lceil \log_a n \rceil$. If S is any n-vertex subset of $H_{d,a}$, then*

$$\rho\big(H_{d,a}[S]\big) \leq (a-1)\log_a n.$$

Equality is possible only when $n = a^d$; in that case equality is attained by the whole Hamming graph.

Proof. Minimality of d gives $a^{d-1} < n \leq a^d$. Set $N = a^d$, $D = d(a-1)$, and $y = n/N$, so $1/a < y \leq 1$. Lemma 1 gives

$$\rho\big(H_{d,a}[S]\big) \leq D - a + ay. \tag{2}$$

It remains to compare this chord with the logarithm. Define

$$f(y) = \frac{a-1}{\log a} \log y - a(y-1).$$

We have $f(1/a) = f(1) = 0$ and $f''(y) = -(a-1)/(y^2 \log a) < 0$. Strict concavity therefore gives $f(y) > 0$ for $1/a < y < 1$, and $f(1) = 0$. Consequently

$$D - a + ay \leq D + \frac{a-1}{\log a} \log y = (a-1) \log_a n,$$

with strict inequality whenever $y < 1$. Combining this with Eq. (2) proves the theorem and its equality statement. □

Corollary 1 (Reeves et al. conjecture)**.** *For every $a \geq 2$ and $n \geq 1$, the bricklayer graph satisfies*

$$\lambda_{n,a} \leq (a-1) \log_a n,$$

with equality if and only if n is a power of a.

Proof. The case $n = 1$ is immediate. For $n \geq 2$, after deleting fixed leading zeroes, $G_{n,a}$ is exactly an induced n-vertex subgraph of $H_{\lceil \log_a n \rceil, a}$. Apply Theorem 1. At $n = a^d$, $G_{n,a} = H_{d,a}$ has spectral radius $d(a-1) = (a-1)\log_a n$. □

For $a = 2$, the same argument supplies a short alternative proof of the binary theorem in Ref. [1]. The restriction to minimal dimension is essential to the stronger statement for arbitrary induced subgraphs: an n-vertex star embeds in a sufficiently high-dimensional hypercube and can have spectral radius exceeding $\log_2 n$.

4 Coordinate-weighted Hamming graphs

We next retain the product Hamming geometry but allow coordinates to mutate at different rates. Let A_j act as the adjacency matrix of K_a in coordinate j and as the identity in every other tensor factor, and define

$$A_w = \sum_{j=1}^{d} w_j A_j, \qquad w_j > 0.$$

Thus an edge changing coordinate j has weight w_j. Put

$$N = a^d, \qquad D = (a-1) \sum_{j=1}^{d} w_j, \qquad w_{\min} = \min_j w_j.$$

Theorem 2 (anisotropic Hamming bound)**.** *For every $S \subseteq [a]^d$ of size n with $a^{d-1} \leq n \leq a^d$,*

$$\rho\big(A_w[S]\big) \leq D + (a-1) w_{\min} \log_a \left(\frac{n}{a^d}\right). \tag{3}$$

The inequality is strict when $a^{d-1} < n < a^d$. At $n = a^d$, equality is attained by the whole state space. At $n = a^{d-1}$, equality holds exactly for sets

$$S = \{x \in [a]^d : x_j = c\},$$

where $w_j = w_{\min}$ and $c \in [a]$.

Proof. The coordinate operators commute. Since the eigenvalues of $A(K_a)$ are $a-1$ on constants and -1 on the mean-zero subspace, the tensor eigenspace indexed by a coordinate set T has eigenvalue

$$D - a \sum_{j \in T} w_j.$$

The largest nonprincipal eigenvalue is therefore $D - aw_{\min}$. Applying Lemma 1 with $g = aw_{\min}$ and $y = n/N$ gives

$$\rho\big(A_w[S]\big) \le D - aw_{\min}(1-y). \tag{4}$$

For $1/a \le y \le 1$, the same strict concavity calculation used in Theorem 1 gives

$$-a(1-y) \le (a-1)\log_a y,$$

with equality only at $y = 1/a$ and $y = 1$. Combining this inequality with Eq. (4) proves Eq. (3), including strictness in the interior.

It remains to classify equality at $y = 1/a$. Equality in Lemma 1 forces a nonnegative Rayleigh maximizer, extended by zero, to be constant on S and its centered component to lie in the $D - aw_{\min}$ eigenspace. Consequently

$$\mathbf{1}_S - \frac{1}{a}\mathbf{1} = \sum_{j:\, w_j = w_{\min}} h_j(x_j), \qquad \sum_{c \in [a]} h_j(c) = 0.$$

The left-hand side takes only two values. If two summands on the right were nonconstant, varying their two coordinates would produce at least three distinct sums. Hence only one summand is nonconstant, so membership in S depends on one minimum-weight coordinate. Since $|S|/N = 1/a$, exactly one symbol of that coordinate belongs to S. Conversely, every such slice is a copy of a $(d-1)$-fold weighted Hamming product with spectral radius $D - (a-1)w_{\min}$, which equals the right side of Eq. (3) at $n = N/a$. □

Corollary 2 (continuous-time killed decay)**.** *For the continuous-time walk generator $Q = A_w - DI$, let κ_S denote the principal decay rate after killing on exit from S. Under the hypotheses of Theorem 2,*

$$\kappa_S \ge (a-1)w_{\min} \log_a\left(\frac{a^d}{n}\right).$$

Proof. The killed generator is $Q[S] = A_w[S] - DI$, so $\kappa_S = D - \rho(A_w[S])$. Apply Eq. (3). □

The dense-support hypothesis cannot simply be dropped. In the binary hypercube Q_{19}, the set consisting of the zero word and the 19 words of Hamming weight one induces the star $K_{1,19}$. Its spectral radius is $\sqrt{19}$, whereas $\sqrt{19} > \log_2 20$. Thus even the unweighted special case of Eq. (3) fails for arbitrary sparse supports in a fixed high-dimensional ambient cube.

5 Extensions and open problems

The first theorem resolves the stated unweighted higher-alphabet conjecture. Theorem 2 extends it to unequal coordinate weights in the dense regime, but not to arbitrary coordinate kernels. More general reversible kernels naturally replace cardinality by stationary mass. Nonsymmetric operators require different methods: ordinary eigenvalues need not control killed survival, and the asymptotic Perron root of a non-normal killed operator need not control short-time survival. Reversiblization and power-based spectral quantities are standard tools in that setting [4, 5], but a sharp finite-time support theorem remains open.

Other distinct open problems include determining extremizers in the sparse regime, classifying near-equality sets when coordinate rates are heterogeneous or nearly tied, and extending the argument to Cartesian products with nonuniform factors.

References

[1] T. Reeves, R. S. Farr, J. Blundell, A. Gallagher, and T. M. A. Fink, "Eigenvalues of neutral networks: interpolating between hypercubes," *Discrete Mathematics* **339** (2016), 1283–1290. doi:10.1016/j.disc.2015.11.011.

[2] V. Nikiforov, "The spectral radius of subgraphs of regular graphs," *Electronic Journal of Combinatorics* **14** (2007), N20. doi:10.37236/1021.

[3] M.-F. Chen, *Eigenvalues, Inequalities, and Ergodic Theory*, Springer, 2005. doi:10.1007/b138904.

[4] J. A. Fill, "Eigenvalue bounds on convergence to stationarity for nonreversible Markov chains, with an application to the exclusion process," *Annals of Applied Probability* **1** (1991), 62–87.

[5] D. Paulin, "Concentration inequalities for Markov chains by Marton couplings and spectral methods," *Electronic Journal of Probability* **20** (2015), no. 79. doi:10.1214/EJP.v20-4039.