SPECTRAL COMPRESSION, OBSTRUCTIONS, AND LOW-DEFECT STABILITY IN Q-ARY HAMMING GRAPHS

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Abstract. For the q-ary Hamming graph $H_{d,q} = K_q^{\square d}$, we study the fixed-order problem of maximizing the adjacency spectral radius of an induced subgraph with $1 + d(q-1)$ vertices, the order of a radius-one Hamming ball. We first prove a Perron-weight coordinate compression: sorting a nonnegative test function within every K_q coordinate fibre preserves its Euclidean norm and does not decrease its adjacency quadratic form. Consequently a coordinatewise down-set maximizer exists. For a down-set of the target order, the number R of off-axis vertices equals exactly the total deficit D of missing axis vertices. This reduction does not make the ball universally optimal. We give an explicit competitor that strictly beats the radius-one ball in $H_{3,q}$ for every $q \geq 3$, and a lower-dimensional-core construction showing failure for every $q + 1 \leq d \leq 2q$. Hence any eventual threshold, if it exists, is at least $2q+1$; in the ternary case it is at least 16. We then determine the first two compressed defect layers. Every defect-one down-set is strictly inferior to the ball for $q, d \geq 3$. Defect two has three off-axis shapes and 21 stable marked orbits. Exact Schur-complement certificates show that all defect-two down-sets lose for $q \geq 4$; the only non-losing ternary cases are two strict families in dimensions three or four and one tie in dimension four. The general problem for defect at least three, and even the existence of an eventual q-ary threshold, remain open.

1 Introduction

The Hamming graph

$$H_{d,q} = \underbrace{K_q \,\square \cdots \square\, K_q}_{d \text{ factors}}$$

has vertex set $[q]^d = \{0, 1, \ldots, q-1\}^d$, with two words adjacent when they differ in exactly one coordinate. Given $S \subseteq [q]^d$, write $H_{d,q}[S]$ for the induced graph and $\rho(S)$ for its adjacency spectral radius. We consider the discrete Faber–Krahn-type question

$$\max\{\rho(S) : S \subseteq [q]^d,\ |S| = 1 + d(q-1)\}. \tag{1}$$

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OpenAI 5.6 Sol developed the mathematical arguments and performed the computational and formal checks described in this preprint. The GloGlo research team posed the problem, established the research direction, and guided the investigation.

The cardinality is that of the radius-one ball

$$B_{d,q} = \{x \in [q]^d : |\operatorname{supp}(x)| \le 1\}$$

about the origin.

The binary version belongs to a well-developed spectral extremal program. Bollobás, Lee and Letzter [2] proved, among other results, that a fixed-radius binary Hamming ball is eventually optimal when the ambient dimension is sufficiently large. Their proof uses coordinate pushes of a Perron vector. Classical q-ary Hamming compressions instead optimize the number of internal edges [3, 1]; edge count and spectral radius are distinct objectives. The first question is therefore whether an appropriate Perron-weight compression survives when each coordinate fibre is a clique rather than an edge.

We determine the consequences of coordinate compression, construct explicit obstructions to radius-one ball extremality, and classify the first two defect layers. The main results are:

1. a coordinatewise Perron compression and an exact defect conservation law $R = D$;
2. an all-q counterexample in dimension three and explicit obstruction intervals in higher dimension;
3. a uniform theorem excluding all defect-one competitors; and
4. a complete, computer-assisted exact classification of defect two.

The global maximizer and the existence of an eventual q-ary threshold remain open.

2 Compression to coordinatewise down-sets

Let A denote the adjacency matrix of $H_{d,q}$. For a nonnegative function $f : [q]^d \to \mathbb{R}_{\ge 0}$, define

$$\mathcal{E}(f) = \langle f, Af \rangle.$$

Fix a coordinate i. For every choice z of the other $d-1$ coordinates, let $f_z = (f_z(0), \dots, f_z(q-1))$ be the corresponding fibre vector. The operator C_i rearranges every such vector into nonincreasing order and places the rearranged entries at symbols $0, 1, \dots, q-1$.

Theorem 1 (Perron-weight coordinate compression)**.** *For every d, q, every coordinate i, and every nonnegative f,*

$$\|C_i f\|_2 = \|f\|_2, \qquad \langle C_i f, AC_i f \rangle \ge \langle f, Af \rangle.$$

Consequently, if $C_i S$ replaces the occupied symbols in each coordinate-i fibre of a set S by the initial segment of the same size, then

$$|C_i S| = |S|, \qquad \rho(C_i S) \ge \rho(S).$$

At every fixed cardinality, a coordinatewise down-set spectral maximizer exists.

Proof. Sorting preserves each fibre's multiset and hence preserves the global ℓ_2 norm. Decompose the quadratic form by edge direction. Along coordinate i, the contribution of a fibre vector v is

$$\sum_{a \ne b} v_a v_b = \left(\sum_a v_a \right)^2 - \sum_a v_a^2,$$

which is invariant under permutation.

For a different coordinate j, fix all coordinates except i, j. If v_b is the coordinate-i vector at j-symbol b, then the contribution from the j-edges in this slice is

$$\sum_{b \neq c} \langle v_b, v_c \rangle.$$

The rearrangement inequality gives $\langle v_b^{\downarrow}, v_c^{\downarrow} \rangle \geq \langle v_b, v_c \rangle$ for every pair. Summing over all pairs, slices, and $j \neq i$ proves the energy inequality.

For the set statement, take a nonnegative Perron vector on a component of $H_{d,q}[S]$ attaining $\rho(S)$ and extend it by zero. After sorting, its positive entries in each fibre lie within the initial segment occupied by $C_i S$. The Rayleigh principle proves $\rho(C_i S) \geq \rho(S)$, without a connectedness assumption.

Coordinate compressions terminate. Indeed, whenever the support changes, the nonnegative integer potential

$$\Phi(S) = \sum_{x \in S} \sum_{k=1}^{d} x_k$$

strictly decreases. A terminal set is invariant under every coordinate compression and is therefore a down-set in the product order. Applying this to a fixed-cardinality maximizer proves existence of a down-set maximizer. $\square$

The proof is the K_q-fibre analogue of a binary Perron-vector push in Ref. [2]; the rearrangement mechanism itself should not be viewed as an unprecedented compression principle.

At the target order, compression exposes an exact accounting parameter. For a down-set S, put

$$a_i = |\{t \in \{1, \ldots, q-1\} : te_i \in S\}|, \quad D = \sum_{i=1}^{d} (q - 1 - a_i),$$

and let R be the number of vertices of S with at least two nonzero coordinates.

Proposition 2 (defect conservation)**.** *If S is a down-set of order $1 + d(q-1)$, then*

$$R = D.$$

Moreover $R = 0$ if and only if $S = B_{d,q}$.

Proof. The origin, the $\sum_i a_i$ axis vertices, and the R off-axis vertices partition S. Therefore

$$1 + d(q-1) = 1 + \sum_i a_i + R,$$

which rearranges to $R = D$. If $R = 0$, then $D = 0$, so every axis has height $q - 1$ and S is exactly the ball. The converse is immediate. $\square$

Thus each off-axis vertex is paid for by one missing axis position. This is a reduction, not an exchange theorem: a cluster of off-axis vertices may gain more spectral mass than the deleted axis positions lose.

3 Explicit obstructions to ball extremality

The ball consists of a root joined to d disjoint cliques K_{q-1}. The root–shell partition is equitable, and hence

$$\beta_{d,q} := \rho(B_{d,q}) = \frac{q-2+\sqrt{(q-2)^2+4d(q-1)}}{2}. \tag{2}$$

Equivalently, $\beta_{d,q}$ is the positive root of

$$x^2-(q-2)x-d(q-1)=0. \tag{3}$$

3.1 An all-q obstruction in dimension three

For $q \geq 3$, define

$$R_q = (\{0,1\}\times[q]) \cup (\{2,\dots,q-1\}\times\{0\}) \subseteq H_{2,q} \subseteq H_{3,q},$$

where the final inclusion fixes the third coordinate.

Proposition 3 (dimension-three counterexample)**.** *For every integer $q \geq 3$,*

$$|R_q| = |B_{3,q}| = 3q-2 \quad \textit{and} \quad \rho(R_q) > \rho(B_{3,q}).$$

More precisely,

$$\rho(B_{3,q}) = \frac{q-2+\sqrt{q^2+8q-8}}{2}, \qquad \rho(R_q) = \frac{q-1+\sqrt{q^2+6q-7}}{2}.$$

Proof. The cardinalities are immediate. Partition R_q into

$$A=\{0,1\}\times\{0\}, \quad C=\{0,1\}\times\{1,\dots,q-1\}, \quad E=\{2,\dots,q-1\}\times\{0\}.$$

This is equitable, with quotient

$$Q_R = \begin{pmatrix} 1 & q-1 & q-2 \\ 1 & q-1 & 0 \\ 2 & 0 & q-3 \end{pmatrix}.$$

Its characteristic polynomial factors as

$$(x-q+2)\{x^2-(q-1)x-2(q-1)\}.$$

The positive quadratic root is the Perron root displayed above. If it is r_q and the ball root is β_q, then

$$2(r_q-\beta_q) = 1+\sqrt{q^2+6q-7}-\sqrt{q^2+8q-8} > 0.$$

Indeed, after squaring positive quantities, the last inequality reduces to $(q-1)^2 < q^2+6q-7$, whose gap is $8(q-1)>0$. □

The proposition gives a witness, not the true maximizer. The same Ferrers shape in two dimensions is related to signless-Laplacian extremal results for bipartite graphs [5]; the proposition is used only as an explicit spectral obstruction.

3.2 Lower-dimensional cores and obstruction intervals

Put $Q = q - 1$. For $k \geq 2$, define

$$a_k = \frac{q^k - 1}{q - 1} = 1 + q + \cdots + q^{k-1}, \qquad c_k = k\{k(q-1) - (q-2)\}. \tag{4}$$

At dimension $d = a_k$, the target order $1 + dQ$ is exactly q^k, so a full $H_{k,q}$ fits. For $d > a_k$, add $d - a_k$ complete new-coordinate arms of Q vertices at the zero word. Denote the resulting connected induced graph by $C_{d,k,q}$.

This construction is feasible because $a_k \geq k$, and therefore $d - a_k \leq d - k$ unused coordinates are available. Fixing every other coordinate makes the union induced in $H_{d,q}$, and its order is

$$|C_{d,k,q}| = q^k + (d - a_k)(q - 1) = 1 + d(q - 1).$$

Theorem 4 (core-and-arm obstruction)**.** *If $a_k < c_k$, then*

$$\rho(C_{d,k,q}) > \rho(B_{d,q}) \qquad \textit{for every integer } a_k \leq d \leq c_k.$$

If $a_k = c_k$, the unaugmented core ties the ball at that dimension; strict inequality is not asserted.

Proof. The core has q^k vertices and is kQ-regular. Every added arm preserves connectedness and makes a proper supergraph, so

$$\rho(C_{d,k,q}) \geq kQ,$$

with strict inequality when $d > a_k$. Evaluating the ball polynomial (3) at kQ gives

$$(kQ)^2 - (q - 2)kQ - dQ = Q(c_k - d).$$

Thus $kQ > \beta_{d,q}$ for $d < c_k$ and $kQ = \beta_{d,q}$ for $d = c_k$. At the upper endpoint, $a_k < c_k$ ensures at least one added arm and therefore strict Perron monotonicity. The remaining cases follow directly. □

Corollary 5 (necessary threshold bounds)**.** *For every $q \geq 3$, the ball is strictly nonmaximal for*

$$q + 1 \leq d \leq 2q.$$

Consequently, if a finite threshold $d_0(q)$ exists such that $B_{d,q}$ is a maximizer for every $d \geq d_0(q)$, then

$$d_0(q) \geq 2q + 1.$$

For $q = 3$, the $k = 3$ core also obstructs $d = 13, 14, 15$, and hence $d_0(3) \geq 16$.

Proof. For $k = 2$, Eq. (4) gives $a_2 = q + 1$ and $c_2 = 2q$. For $(q, k) = (3, 3)$ it gives $a_3 = 13$ and $c_3 = 15$. Apply Theorem 4. □

These are lower bounds on a hypothetical eventual threshold, not a proof that the threshold exists. At $(q, k, d) = (4, 3, 21)$, one has $a_k = c_k$ and both the core and the ball have spectral radius 9, illustrating why the endpoint hypothesis in Theorem 4 is essential.

4 Defect-one stability

Compression and Proposition 2 make R a natural measure of distance from the ball. The first nonzero layer is completely stable.

Theorem 6 (defect one)**.** *Let $q \geq 3$, $d \geq 3$, and let $S \subseteq [q]^d$ be a coordinatewise down-set of order $1 + d(q-1)$. If $R = 1$, then*

$$\rho(S) < \rho(B_{d,q}).$$

Proof. Write $Q = q-1$, $p = q-2$, $\beta = \beta_{d,q}$, and

$$s = \beta - p > 0.$$

Then $s\beta = dQ$. Since $d \geq 3$ and $q \geq 3$, monotonicity of $x(x+p)$ and

$$2(2+q-2) = 2q \leq d(q-1) = s(s+q-2)$$

show that $s \geq 2$. We will also use

$$s\beta = d(q-1) \geq 6, \qquad \beta - \frac{2}{s} > 0.$$

The unique off-axis vertex must be $e_i + e_j$. A coordinate value at least two or support of size at least three would force a second off-axis predecessor by downward closure. Since $R = D = 1$, exactly one axis clique is shortened from K_Q to K_p. Up to symmetry, the shortened arm is either incident to the new point or is uninvolved.

Delete the root. For a clique K_r and $\beta > r - 1$,

$$(\beta I - A_{K_r})^{-1} = \frac{1}{\beta+1} I + \frac{1}{(\beta+1)(\beta - r + 1)} J. \tag{5}$$

Thus the root self-energy of an intact arm is Q/s and that of the shortened arm is $p/(s+1)$. The exact loss is

$$L = \frac{Q}{s} - \frac{p}{s+1} = \frac{s+Q}{s(s+1)}. \tag{6}$$

Now add the off-axis vertex z, joined to one chosen vertex in each of two arms. If $R_i = (\beta I - A_i)^{-1}$ is the resolvent of an involved arm, u_i is its all-ones root-incidence vector, and c_i selects the attachment vertex, a second Schur complement gives the exact gain

$$G = \frac{(u_1^T R_1 c_1 + u_2^T R_2 c_2)^2}{\beta - (c_1^T R_1 c_1 + c_2^T R_2 c_2)}.$$

Equation (5) gives $u_i^T R_i c_i = 1/s$ for a full arm and $1/(s+1)$ for the shortened arm; positivity gives $c_i^T R_i c_i \leq u_i^T R_i c_i$. Uniformly in the two geometries,

$$G \leq \frac{4}{s(s\beta - 2)}. \tag{7}$$

Finally, Eqs. (6)–(7) satisfy $G < L$. After clearing positive denominators, it is enough to show

$$4(s+1) < (s+Q)\{s(s+p) - 2\}.$$

For $s \geq 2$ and $p \geq 1$, the right-hand side is at least $(s+2)^2(s-1)$, and

$$(s+2)^2(s-1) - 4(s+1) = s^3 + 3s^2 - 4s - 8 > 0.$$

The last polynomial equals 4 at $s = 2$ and is increasing thereafter. The root Schur complement at β is therefore positive, equivalently $\beta I - A_S$ is positive definite. Hence $\rho(S) < \beta$. $\square$

The theorem is local in defect. It cannot be iterated blindly: defect-two sets can beat the ball even though every defect-one set loses.

5 Exact classification of defect two

Let S be a down-set of target order with $R = D = 2$. The off-axis part is a two-element order ideal among vertices with at least two nonzero coordinates.

Lemma 7 (three off-axis shapes)**.** *Up to coordinate permutations, the two off-axis vertices have exactly one of the following forms:*

$$\begin{array}{lll} \mathrm{W}: & \{e_1+e_2, e_1+e_3\}, & \textit{wedge}, \\ \mathrm{M}: & \{e_1+e_2, e_3+e_4\}, & \textit{matching}, \\ \mathrm{C}: & \{e_1+e_2, 2e_1+e_2\}, & \textit{chain}. \end{array}$$

Proof. If both elements are minimal, they are binary weight-two atoms. Their supports either intersect in one coordinate or are disjoint, giving W or M. If the upper element is not minimal, support size at least three would force at least three atoms below it. It therefore has two positive coordinates, say values r, t, whose off-axis principal ideal contains rt elements. Since there are exactly two, $\{r, t\} = \{1, 2\}$, giving C. □

The deficit pattern is either two missing positions on one coordinate, denoted 2, or one on each of two coordinates, denoted 11. Marking these deficits up to the stabilizer of W, M, or C gives the orbit table below. For W, C is the wedge center, L a leaf and O an outside coordinate. For M, E is a matching endpoint and O is outside; P and X distinguish endpoints in the same and in different matched pairs. For the chain, H is the height-two coordinate, L the other involved coordinate and O an outside coordinate.

Table 1: The 21 stable marked defect-two orbits and their minimum parameters.

W orbit	min. (d,q)	M orbit	min. (d,q)	C orbit	min. (d,q)
W-2C	$(3,4)$	M-2E	$(4,4)$	C-2H	$(3,5)$
W-2L	$(3,4)$	M-2O	$(5,3)$	C-2L	$(3,4)$
W-2O	$(4,3)$	M-11P	$(4,3)$	C-2O	$(3,3)$
W-11CL	$(3,3)$	M-11X	$(4,3)$	C-11HL	$(3,4)$
W-11CO	$(4,3)$	M-11EO	$(5,3)$	C-11HO	$(3,4)$
W-11LL	$(3,3)$	M-11OO	$(6,3)$	C-11LO	$(3,3)$
W-11LO	$(4,3)$			C-11OO	$(4,3)$
W-11OO	$(5,3)$				

Theorem 8 (defect-two classification)**.** *Let $q \geq 3$, $d \geq 3$, and let $S \subseteq [q]^d$ be a coordinatewise down-set of order $1 + d(q-1)$ with $R = 2$.*

1. *If $q \geq 4$, then $\rho(S) < \rho(B_{d,q})$.*
2. *If $q = 3$, every feasible orbit loses strictly except the following:*

$$\begin{array}{rcl} (d, \textit{orbit}) = (3, \text{C-2O}) & : & \rho(S) > \rho(B_{d,q}), \\ (d, \textit{orbit}) = (4, \text{C-2O}) & : & \rho(S) > \rho(B_{d,q}), \\ (d, \textit{orbit}) = (3, \text{C-11LO}) & : & \rho(S) > \rho(B_{d,q}), \\ (d, \textit{orbit}) = (4, \text{C-11OO}) & : & \rho(S) = \rho(B_{d,q}). \end{array}$$

Proof. We give the exact finite-certificate reduction. It is useful both as a proof and as a reproducibility specification.

Put $\beta = \beta_{d,q}$ and $s = \beta - (q-2)$, so $s\beta = d(q-1)$ and $s \geq 2$. After deleting the root, the axis graph is

$$C = \bigsqcup_{i=1}^{d} K_{q-1-\delta_i}, \qquad \sum_i \delta_i = 2.$$

Let a be the all-ones vector on its axis vertices, W the two off-axis vertices, B the axis-to-W incidence matrix, and A_W the adjacency matrix inside W. Thus $A_W = 0$ for W and M and $A_W = A(K_2)$ for C. Define

$$D_\beta = \beta I - A_C, \quad u = B^T D_\beta^{-1} a, \quad T = \beta I - A_W - B^T D_\beta^{-1} B.$$

For an arm of deficit δ, Eq. (5) has $\beta - r + 1 = s + \delta$. It follows exactly that an attachment vertex e on that arm satisfies

$$a^T D_\beta^{-1} e = \frac{1}{s+\delta},$$

and, for attachment vertices e, f,

$$e^T D_\beta^{-1} f = \begin{cases} \dfrac{s+\delta+1}{(\beta+1)(s+\delta)}, & e = f, \\ \dfrac{1}{(\beta+1)(s+\delta)}, & e \neq f \text{ on one arm}, \\ 0, & \text{on distinct arms.} \end{cases}$$

Thus u and the 2×2 matrix T are explicit rational functions of s, q for every row of Table 1.

The exact root-resolvent loss from shortening the axes is

$$L = \sum_i \left(\frac{q-1}{s} - \frac{q-1-\delta_i}{s+\delta_i} \right) = \sum_i \frac{\delta_i(s+q-1)}{s(s+\delta_i)}, \tag{8}$$

while adding the off-axis pair restores

$$G = u^T T^{-1} u. \tag{9}$$

Each off-axis vertex attaches to two axis vertices, and each corresponding diagonal resolvent contribution is at most $1/s$. Hence

$$T_{ii} \geq \beta - \frac{2}{s} > 0,$$

because $s\beta = d(q-1) \geq 6$. Direct exact expansion shows that, after substituting $q = 3 + Q$ and $s = 2 + z$, the determinant numerator of T has nonnegative integer coefficients and a positive constant in every feasible orbit. The leading-minor criterion therefore gives $T \succ 0$. Two Schur complements now give

$$\operatorname{sign}(\beta - \rho(S)) = \operatorname{sign}(L - G). \tag{10}$$

For $q \geq 4$, clear the positive denominator of $L - G$, remove the common positive factor $q + s - 1$, and substitute $q = 4 + Q$, $s = 2 + z$. For each of the 21 orbits the resulting integer polynomial has only nonnegative coefficients and a positive constant. This proves strict positivity for all $Q, z \geq 0$.

For $q = 3$, use $s(s+1) = 2d$. Every feasible orbit is manifestly positive after the same shift except four boundary expressions. Up to positive factors, they are

$$\begin{array}{ll} \text{W-2O}: & s^3 + 3s^2 - 5s - 11, \quad \text{positive for feasible } d \geq 4, \\ \text{C-2O}: & s^2 - s - 4, \quad \text{negative for } d = 3, 4 \text{ and positive for } d \geq 5, \\ \text{C-11LO}: & s^4 + 3s^3 - 5s^2 - 11s - 3, \quad \text{negative only for } d = 3, \\ \text{C-11OO}: & s^2 + s - 8 = 2d - 8, \quad \text{zero only for } d = 4. \end{array}$$

The signs, together with Eq. (10), give precisely the stated classification. The 21 exact coefficient lists and independent full-matrix checks are included in the accompanying reproducibility archive described in Appendix A. □

The successful defect-two exceptions are chains; adjacency and arm reuse suggest the mechanism exposed by the resolvent calculation. Wedges and matchings never beat the ball. This also shows why defect is not a monotone notion of spectral disadvantage: in the smallest ternary cases, defect two wins whereas defect one always loses.

6 What is resolved and what remains open

The results separate three logically different statements.

First, compression is global: every arbitrary support has a down-set image of the same order and no smaller spectral radius. Second, the obstruction theorems are existential: they prove that balls lose at specified dimensions but do not identify the maximizing support. Third, the low-defect results are local stability theorems inside the compressed class. They do not bound defect $R \geq 3$.

The natural remaining question is therefore:

Conjecture 9 (eventual radius-one extremality)**.** *For every fixed $q \geq 3$, there exists $d_0(q) < \infty$ such that the radius-one ball $B_{d,q}$ maximizes adjacency spectral radius among all induced $1+d(q-1)$-vertex subgraphs of $H_{d,q}$ for every $d \geq d_0(q)$.*

The conjecture is stated as a target, not a result. Corollary 5 forces $d_0(q) \geq 2q+1$ and $d_0(3) \geq 16$ if it exists. Theorems 6 and 8 show that an eventual counterexample for $q \geq 4$ must have defect at least three. What is missing is a uniform higher-rank resolvent or exchange inequality that controls the interaction of an arbitrary off-axis ideal with the locations of the deleted axis positions.

The problem is distinct from the higher-alphabet bricklayer inequality of Reeves et al. [4]. That problem concerns a specified lexicographic family, or in a strengthened form supports in the minimal dimension capable of containing them. Here the support size grows with the ambient dimension, competitors are unrestricted induced subsets before compression, and the question is which geometry maximizes the principal eigenvalue at the order of a radius-one ball.

7 Relation to prior work

The binary compression precedent and eventual ball theorem are due to Bollobás, Lee and Letzter [2]. Lindsey [3] and Ahlswede–Cai [1] are prior art for q-ary Hamming edge isoperimetry, a different objective. Reeves et al. [4] provide an earlier spectral framework for induced subgraphs of Hamming graphs. The results here concern adjacency spectral radius rather than edge count and combine a q-ary Perron compression with explicit obstruction families and exact low-defect classifications.

8 Lean formalization

A companion Lean 4 project is provided in `formal-math/qary_spectral`. Its layout follows a challenge/solution release pattern: `Challenge.lean` states the declarations, `Solution.lean` supplies proofs, `comparator.json` fixes the comparison surface, `scripts/PrintAxioms.lean` reports dependencies, and the project pins its Lean and Mathlib versions. The repository also contains an audit record and continuous-integration configuration.

The current formalization kernel-checks the named algebraic and finite-cardinality declarations corresponding to the radius-one quadratic, its nonnegative root, the strict dimension-three radical comparison, the lower-dimensional-core polynomial and threshold identities, the core cardinality and parameter specializations, and the defect-conservation identity. The graph-theoretic Perron compression, the spectral-radius comparison connecting the displayed algebra to induced Hamming subgraphs, the defect-one resolvent argument, and the exhaustive defect-two orbit classification remain paper proofs. The comparator declaration list in the companion project enumerates the kernel-checked statements.

A Reproducibility archive

The derivations were independently audited from raw definitions. From the repository root, the principal exact checks are:

```
.venv/bin/python validation/qary_compression_stability_audit/check_claims.py
.venv/bin/python validation/qary_radius_one_dimension_threshold_audit/
  check_obstructions.py
.venv/bin/python validation/qary_defect_one_audit/check_defect_one.py
python3 validation/qary_defect_two_audit/checker.py
.venv/bin/python validation/qary_defect_two_audit/symbolic_cert.py
```

The defect-two supplement released with this preprint is version `2026-08-29-v1`. It contains the rational formulas for every marked orbit, positive-denominator checks, coefficient-positivity certificates, exact characteristic polynomials for boundary cases, and independent finite full-adjacency comparisons. Floating-point eigenvalues are corroborative only; the infinite statements rest on the displayed algebra and exact integer coefficient certificates.

File	SHA-256
`checker.py`	`cc2ce44b3b7cdd20473456bd22da14241` `fba060284caebce694037af2cd8e112`
`results.json`	`b1e5fb1f54b30004ec0cbfe268244564` `53ca525ab249c0e8634e63a7c34de4b1`
`symbolic_cert.py`	`dca4161e583e48889e79e226536807b9` `112faa486886b5d8648a985c37794b59`
`symbolic_certificates.json`	`ce23b050cdd6f626d882889a6839dfdf` `2da51e7b3994409afdb4c3817417a34a`

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